



Approximate solution of the two-dimensional Rayleigh-Stokes problem for a heated generalized second grade fluid with fractional derivatives

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Abstract

In this paper, a scheme based on Sinc and radial basis functions (RBF) is developed to approximate the solution of two-dimensional Rayleigh-Stokes problem for a heated generalized second-grade fluid with fractional derivatives. We use RBF and Sinc functions as basis functions to approximate spatial and time coordinates of the unknown function, respectively. The error analysis is studied and discussed. An illustrative example verifies the effectiveness of our method and shows that one can obtain accurate results with only a small number of basis functions.

Introduction

Models of fractional equations have been used in many fields during recent years. Historical and theoretical aspects of fractional calculus were studied in [15,6]. Also, There are many textbooks related to applications of fractional equations in various fields of science and engineering, such as Control theory, Biology, Chemistry, Engineering, Physics and Continuum Mechanics. [2,5,10,13,18,20]

Here we consider that the two-dimensional Rayleigh-Stokes problem with fractional derivative for heated generalized second-grade fluid [13]

$$\frac{\partial u(x, y, t)}{\partial t} = {}_0D_t^{1-\gamma} \left[\frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} \right] + \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} + f(x, y, t), \quad 0 < t \leq T, \quad (x, y) \in \Omega, \quad (1)$$

with boundary conditions

$$u(x, y, t) = w_1(x, y, t), \quad (x, y) \in \partial\Omega, \quad (2)$$

and initial condition

$$u(x, y, 0) = w_2(x, y), \quad (x, y) \in \Omega, \quad (3)$$

where $f(x, y, t)$ is known function, $\Omega = [a, b] \times [c, d]$, $0 < \gamma < 1$ and ${}_0D_t^{1-\gamma} u(x, y, t)$ is the Riemann-Liouville fractional derivative of order $1 - \gamma$ which is defined by

$${}_0D_t^{1-\gamma}u(x, y, t) = \frac{\partial}{\partial t} {}_0I_t^\gamma u(x, y, t), \tag{4}$$

where ${}_0I_t^\gamma$ is the fractional integral operator,

$${}_0I_t^\gamma u(x, y, t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{u(x, y, \tau)}{(t - \tau)^{1-\gamma}} d\tau. \tag{5}$$

The Rayleigh-Stokes problem is a model of non-Newtonian behavior exhibited by certain fluids, the flow characteristics of non-Newtonian viscoelastic fluids through a dual porous medium [21] and the flow analysis of fluids in fractal reservoir with fractional derivative studied in [26] by using the Hankel transform, the discrete Laplace transform of sequential fractional derivatives and the generalized Mittag-Leffler function. The Fourier sine transform and the Laplace transform were used for the solution of problem (1)-(3) in [11,22,23].

Several methods are applied to approximate the solution of the Rayleigh-Stokes problem (1)-(3), such as the explicit and implicit finite difference methods [3,4, 28], a high order difference scheme and RBF mesh less method [14].

The implicit and explicit finite difference methods for the variable-order Rayleigh-Stokes problem for a heated generalized second grade fluid with fractional derivative is given in [29].

Okayama [16] developed two new Sinc schemes based on single and double exponential transformation for Caputo's fractional derivatives of order $0 < \gamma < 1$. They used these methods to solve linear Fredholm integral equations of the second kind with weakly singular kernel [17]. In fact, the authors in [16,17,27] took the idea that was presented by Riley [19], to develop the techniques in Sinc methods to approximate the solution of the second kind weakly singular linear Volterra integral equations.

Baumann and Stenger [1] provided a survey on application of Sinc methods to solve fractional integral, fractional derivatives, fractional equations and fractional differential equations.

In this study, we approximate the solution of problem (1)-(3) by using sinc-RBF collocation method. In section 2, we review the RBFs approximation method. In section 3, some properties of Sinc function are given. In section 4, we develop the Sinc-RBF collocation method, and the error analysis of the proposed method is given. Finally, in section 5, an illustrative example is given.

RBFs approximation method

The radial basis functions (RBFs) are functions that depend on the distance from central points, so one can use RBFs to reduce higher dimensional space problem to lower dimensions [7, 8,9].

The approximate expansion of $u(X)$ can be obtained by

$$u(X) = \sum_{i=1}^N d_i \varphi(\|X_i - X\|_2) = \sum_{i=1}^N d_i \varphi_i(r), \tag{6}$$

where $X_i, i = 1, 2, \dots, N$ are center points, the $\|\cdot\|_2$ is the Frobenius norm, d_i are unknown coefficients and φ are RBF functions. There are several kinds of RBFs, such as Thin plate splines(TPS)

$\varphi(r) = r^{2\beta} \log(r), \beta \in N$), Multiquadrics(MQ) ($\varphi(r) = (r^2 + c^2)^{\frac{1}{2}}$), Inverse Multiquadrics(IMQ)

($\varphi(r) = (r^2 + c^2)^{-\frac{1}{2}}$), Gaussians(GAU) ($\varphi(r) = e^{-\frac{r^2}{c^2}}$), Odd degree splines ($\varphi(r) = r^\beta, \beta > 0, \beta \notin 2N$),

etc, where c is the shape parameter which takes arbitrary values. The Multiquaric radial basis function was introduced for solution of partial differential equations by Kansa. The exponential convergence of RBF was studied by [8,9,29]. Here we use Multiquadrics basis function.

Sinc function

In this section, we review some properties of Sinc function, Sinc interpolation and Sinc quadrature [12,24,25] that we need. The Sinc function is defined by

$$\text{Sinc}(t) = \begin{cases} \frac{\sin(\pi t)}{\pi t}, & t \neq 0 \\ 1, & t = 0 \end{cases}.$$

Let j be an integer and h be a positive number, the j -th shifted Sinc function is defined by

$$S(j, h)(t) = \text{Sinc}\left(\frac{t - jh}{h}\right).$$

Since significant of problems are defined over a finite interval (a, b) , we need the single exponential (SE) transformation that maps a finite interval (a, b) to $(-\infty, +\infty)$ [12,24,25], as follows

$$t = \psi_{a,b}(z) = \frac{b-a}{2} \tanh\left(\frac{z}{2}\right) + \frac{b+a}{2},$$

and its inverse functions defined by

$$z = (\psi_{a,b})^{-1}(t) = \phi_{a,b}(t) = \log\left(\frac{t-a}{b-t}\right),$$

that we can define Sinc points as $t_k = \psi_{a,b}(kh)$. Following [12], we have

$$\delta_{jk}^{(0)} = \left[S(j, h) \circ (\phi_{a,b}(t)) \right] \Big|_{t=t_k} = \text{Sinc} \left(\frac{\log\left(\frac{\psi_{a,b}(kh)-a}{b-\psi_{a,b}(kh)}\right) - jh}{h} \right) = \begin{cases} 1, & j = k \\ 0, & j \neq k \end{cases}, \quad (7)$$

If a given function f defined on (a, b) , and $f(a)=f(b)=0$ can be presented by truncated cardinal function in the following form [12]

$$f(t) = C_M(f, h)(t) = \sum_{j=-M}^M f(\psi_{a,b}(jh)) S(j, h) \circ (\phi_{a,b}(t)).$$

Definition 3.1. [25] Let D be a simple connected domain and $(a, b) \subset D$ and $\beta > 0$. The family of all analytic functions on D denote by $L_\beta(D)$, and satisfy in $|f(z)| \leq k \left| ((z-a)(z-b))^\beta \right|$ for all $z \in D$ and a positive constant k .

Let $f(t)$ be the analytic function on a strip domain $D_d = \{z \in C : |\text{Im}(z)| < d\}$ for some $d > 0$, and should be bounded in some sense.

When incorporating with SE transformation, the condition should be considered in the translated domain

$$\psi_{a,b}(D_d) = \left\{ z \in C : \left| \arg\left(\frac{z-a}{b-z}\right) \right| < d \right\},$$

From [24], if $f \in L_\beta(\psi_{a,b}(D_d))$ for $0 < d < \pi$ and interpolated by SE transformation of Sinc, then there exists constant $k_1 > 0$ independent of M , such that

$$\max_{a < t \leq b} \left| f(t) - \sum_{j=-M}^M f(\psi_{a,b}(jh)) S(j, h) \circ (\phi_{a,b}(t)) \right| \leq k_1 \sqrt{M} e^{-\sqrt{\pi d} \beta M}, \quad (8)$$

where $h = \sqrt{\frac{\pi d}{\beta M}}$.

The truncated Sinc quadrature rule is defined by

$$\int_a^b f(t) dt = h \sum_{j=-M}^M f(\psi_{a,b}(jh)) \psi'_{a,b}(jh), \quad (9)$$

From [16], if $(f / \phi'_{a,b}) \in L_\beta(\psi_{a,b}(D_d))$ for $0 < d < \pi$ then there exists constant $k_2 > 0$ independent of M , such that

$$\left| \int_a^b f(t) dt - h \sum_{j=-M}^M f(\psi_{a,b}(jh)) \psi'_{a,b}(jh) \right| \leq k_2 e^{-\sqrt{\pi d} \beta M}. \tag{10}$$

In general, the function $f(t)$, with initial condition $f(a)=0$, may not vanish at end point $t=b$, then using the Sinc basis function we have to represent the basis function as

$$\chi_j(\phi_{a,b}(t)) = \begin{cases} S(j, h) \rho(\phi_{a,b}(t)), & j = -M, \dots, M-1 \\ \frac{\rho(t)}{1 + \rho(t)}, & j = M \end{cases},$$

where $\rho(t) = e^{\phi_{a,b}(t)}$. This type of Sinc basis is chosen to satisfy the interpolation at the boundaries. For a given vector $c = [c_{-M}, \dots, c_M]^T$ we approximate the function $f(t)$ by

$$f(t) \approx C_M[y](t) = \sum_{j=-M}^M c_j \chi_j(\phi_{a,b}(t)). \tag{11}$$

Sinc-RBF collocation method

In this section, combination of the Sinc function and multiquadrics basis function is used to approximate solution of problem (1).

By changing variable $u(x, y, t) = U(x, y, t) + w_2(x, y)$, the problem (1)-(3) can be given as

$$\begin{aligned} \frac{\partial U(x, y, t)}{\partial t} = {}_0D_t^{1-\gamma} & \left[\frac{\partial^2 U(x, y, t)}{\partial x^2} + \frac{\partial^2 U(x, y, t)}{\partial y^2} \right] \\ & + \frac{\partial^2 U(x, y, t)}{\partial x^2} + \frac{\partial^2 U(x, y, t)}{\partial y^2} + F(x, y, t), \quad 0 < t \leq T, \quad (x, y) \in \Omega, \end{aligned} \tag{12}$$

with boundary and initial conditions

$$\begin{aligned} U(x, y, t) &= W(x, y, t), \quad (x, y) \in \partial\Omega, \\ U(x, y, 0) &= 0, \quad (x, y) \in \Omega, \end{aligned} \tag{13}$$

where

$$F(x, y, t) = f(x, y, t) + \frac{\partial^2 w_2}{\partial x^2} + \frac{\partial^2 w_2}{\partial y^2}, \quad W(x, y, t) = w_1(x, y, t) - w_2(x, y, t).$$

The step size of spatial variables and grade points is defined by

$$h_x = \frac{b-a}{n-1}, \quad h_y = \frac{d-c}{n-1}, \quad x_p = (p-1)h_x, \quad y_q = (q-1)h_y. \tag{14}$$

By using (6) and (11) the solution of equation (12) can be approximated by

$$U(x, y, t) = \sum_{j=-M}^M d_{ij} \phi_i(r) \chi_j(\phi_{0,T}(t)), \tag{15}$$

where $d_{ij}, i = 1, 2, \dots, N, j = -M, \dots, M, M \in N$ are unknown coefficients and

$$\phi_i(r) = \sqrt{(x - x_p)^2 + (y - y_q)^2 + c^2}, \quad p, q = 1, 2, \dots, n, \quad N = n^2. \tag{16}$$

Using the SE-Sinc approach, the integral part of fractional derivative in (5) can be approximated. Now by changing variable $s = \psi_{0,t}(\tau)$, the given interval $(0, t)$ can be transformed to $(-\infty, \infty)$, then the integral ${}_0I_t^\gamma[g](t)$ for a given function $g(t)$ can be written in the following form

$${}_0 I_t^\gamma [g](t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{g(s)}{(t-s)^{1-\gamma}} ds = \frac{1}{\Gamma(\gamma)} \int_{-\infty}^{\infty} \frac{g(\psi_{0,t}(\tau))(\psi_{0,T})'(\tau)}{(t-\psi_{0,t}(\tau))^{1-\gamma}} d\tau = \frac{t^\gamma}{\Gamma(\gamma)} \int_{-\infty}^{\infty} \frac{g(\psi_{0,t}(\tau))}{(1+e^{-\tau})(1+e^\tau)^\gamma} d\tau, \quad (17)$$

and by applying Sinc quadrature rule (9) we obtain

$${}_0 I_t^\gamma [g](t) \approx \mathfrak{S}_M [g](t) = \frac{t^\gamma}{\Gamma(\gamma)} h \sum_{k=-M}^M \frac{g(\psi_{0,t}(kh))}{(1+e^{-kh})(1+e^{kh})^\gamma}, \quad (18)$$

where $h = \sqrt{\frac{\pi d}{\beta M}}$.

Denoting $\Delta U(x, y, t) = \frac{\partial^2 U(x, y, t)}{\partial x^2} + \frac{\partial^2 U(x, y, t)}{\partial y^2}$ and applying the operator ${}_0 I_t^\gamma$ defined on (18) and

also by using series (15), we can approximate the fractional derivative of (12) by

$$\begin{aligned} {}_0 D_t^{1-\gamma} (\Delta U(x, y, t)) &= \frac{d}{dt} ({}_0 I_t^\gamma [\Delta U](x, y, t)) \approx \frac{d}{dt} (\mathfrak{S}_M [\Delta U](x, y, t)) \\ &= \frac{h}{\Gamma(\gamma)} \frac{d}{dt} \left[t^\gamma \sum_{i=1}^N \sum_{j=-M}^M \sum_{k=-M}^M \frac{d_{ij} \Delta \phi_i(r) \chi_j(\phi_{0,T}(\psi_{0,t}(kh)))}{(1+e^{-kh})(1+e^{kh})^\gamma} \right], \end{aligned} \quad (19)$$

where $\Delta \phi_i(r) = \frac{\phi_i^2(r) + c^2}{\phi_i^3(r)}$.

For convenience, denoting $\phi_{0,T}(\psi_{0,t}(kh))$ by $\Psi_{0,k}(t)$ then

$${}_0 D_t^{1-\gamma} (\Delta U(x, y, t)) = \frac{h}{\Gamma(\gamma)} \sum_{i=1}^N \sum_{j=-M}^M \sum_{k=-M}^M \frac{d_{ij} \Delta \phi_i(r) (\gamma^{\gamma-1} \chi_j(\Psi_{0,k}(t)) + t^\gamma (\Psi_{0,k})'(t) \chi_j'(\Psi_{0,k}(t)))}{(1+e^{-kh})(1+e^{kh})^\gamma}. \quad (20)$$

Now by substituting (20) in equation (12) and using collocation points

$$r_l = (x_{p'}, y_{q'}), \quad t_z = \psi_{0,T}(zh), \quad p', q' = 1, 2, \dots, n, \quad z = -M, \dots, M, \quad (21)$$

we have

$$\begin{aligned} &\sum_{i=1}^N \sum_{j=-M}^M d_{ij} \phi_i(r_l) (\phi_{0,T})'(t_z) \chi_j'(\phi_{0,T}(t_z)) - \\ &\frac{h}{\Gamma(\gamma)} \sum_{i=1}^N \sum_{j=-M}^M \sum_{k=-M}^M \frac{d_{ij} \Delta \phi_i(r_l) (\gamma^{\gamma-1} \chi_j(\Psi_{0,k}(t_z)) + t_z^\gamma (\Psi_{0,k})'(t_z) \chi_j'(\Psi_{0,k}(t_z)))}{(1+e^{-kh})(1+e^{kh})^\gamma} \\ &- \sum_{i=1}^N \sum_{j=-M}^M d_{ij} \Delta \phi_i(r_l) \chi_j(\phi_{0,T}(t_z)) = F(r_l, t_z), \quad p', q' = 2, 3, \dots, n-1, \quad z = -M, \dots, M, \end{aligned} \quad (22)$$

substituting (16) in boundary condition (13) we have

$$\sum_{i=1}^N \sum_{j=-M}^M d_{ij} \phi_i(r_l) \chi_j(\phi_{0,T}(t_z)) = W(x_{p'}, y_{q'}, t_z), \quad (x_{p'}, y_{q'}) \in \partial\Omega, \quad p', q' = 1, n, \quad z = -M, \dots, M. \quad (23)$$

where $x_{p'} = (p'-1)h_x$ and $y_{q'} = (q'-1)h_y$.

Method (22) associated with boundary conditions (23) yield the linear system $N \times (2M + 1)$ equations and $N \times (2M + 1)$ unknown coefficients $d_{ij}, i = 1, 2, \dots, N, j = -M, \dots, M$, by solving this system and substituting unknown coefficients in (15) we can approximate the solution of equations (1)-(3).

4.1. The error analysis of the Sinc-RBF method

Here by assuming $U(x, y, t) = U(X, t) = V(X)G(t)$ and denoting $\phi(t) = \phi_{0,T}(t)$ and $\psi(jh) = \psi_{0,T}(jh)$, we can obtain

$$V(X) = \sum_{i=1}^N a_i \varphi_i(r), \tag{24}$$

$$G(t) = \sum_{j=-M}^M b_j \chi_j(\phi(t)), \tag{25}$$

then by using (15) we have

$$U(X, t) = V(X)G(t) = \sum_{i=1}^N \sum_{j=-M}^M a_i b_j \varphi_i(r) \chi_j(\phi(t)) = \sum_{i=1}^N \sum_{j=-M}^M d_{ij} \varphi_i(r) \chi_j(\phi(t)),$$

Following [16], if suppose $(G / \phi') \in L_\beta(\psi(D_d))$ for $d \in (0, \pi)$ let $\mu = \min\{\gamma, \beta\}$, M is a positive

integer, and $h = \sqrt{\frac{\pi d}{\mu M}}$, then there exists a constant $K_3 > 0$ independent of M such that

$$\left\| {}_0D_t^{1-\gamma} [G](t) - \frac{d}{dt} \mathfrak{S}_M [C_M [G]](t) \right\|_\infty \leq K_3 M e^{-\sqrt{\pi d} \mu M}. \tag{26}$$

First of all we need to prove the following lemma to be used in the main theorem of this section.

Lemma 4.1. Let $h > 0$, then

$$\sup_{t \in (0, T]} \left| \sum_{j=-M}^M b_j \chi_j(\phi(t)) \right| \leq 2K_4 \left(\frac{2}{\pi} (3 + \log(M)) + 1 \right).$$

Proof: By substituting $t = \psi(jh)$ in equation (25) and considering (7) we obtain

$$b_j + \frac{e^{jh}}{1+e^{jh}} b_M = G(\psi(jh)), \quad j = -M, \dots, M-1, \tag{27}$$

using Kronecker δ for $j=M$ we have

$$\frac{e^{Mh}}{1+e^{Mh}} b_M = G(\psi(Mh)), \tag{28}$$

by substituting (28) into (27) and assuming $\max_{t \in (0, T]} |G(t)| \leq K_4$, for $j < M$ we obtain

$$|b_j| \leq K_4 \left(1 + \frac{1+e^{Mh}}{e^{Mh}} \frac{e^{jh}}{1+e^{jh}} \right) = K_4 \left(1 + \frac{1+e^{-Mh}}{1+e^{-jh}} \right) \leq 2K_4,$$

and for $j=M$ we have

$$|b_M| \leq |G(\psi(Mh))| \frac{1+e^{Mh}}{e^{Mh}} \leq 2K_4,$$

following [17] we have

$$\sup_{t \in (0, T]} \sum_{j=-M}^M |S(j, h) \circ(\phi(t))| \leq \frac{2}{\pi} (3 + \log(M)),$$

since $\frac{e^{\phi(t)}}{1+e^{\phi(t)}} \leq 1$, then we can obtain

$$\sup_{t \in (0, T]} \sum_{j=-M}^M |\chi_j(\phi(t))| \leq \frac{2}{\pi} (3 + \log(M)) + 1. \tag{29}$$

The proof is completed.

Theorem 4.1. Suppose that $\varphi = \varphi(\|\cdot\|)$ is a conditionally positive definite function, and let $V(X)$ interpolated by φ and satisfied $|V^{(l)}(r)| \leq l! K_5^l$ for all $r \in [0, \infty)$ and $1 \leq l_0$, and $K_5 > 0$. Then there exist positive constants K_6 , K_7 and K_8 such that

$$\max_{X=(x,y) \in [a,b]^2} \left| V(X) - \sum_{i=1}^N a_i \varphi_i(r) \right| \leq K_6 e^{-\frac{K_8}{h}}, \tag{30}$$

and

$$\max_{\substack{X=(x,y) \in [a,b]^2 \\ t \in (0,T]}} \left| V(X) G(t) - \sum_{i=1}^N \sum_{j=-M}^M a_i b_j \varphi_i(r) \chi_j(\phi(t)) \right| \leq K_7 \sqrt{M} e^{-K_8 \sqrt{M^*}}, \tag{31}$$

where $M^* = \min\{M, N\}$.

Proof: Following [30, Theorem 11.22, page 190] the relation (30) can be proved. Easily by considering

$$\begin{aligned} & \max_{\substack{X \in [a,b]^2 \\ t \in (0,T]}} \left| V(X) G(t) - \sum_{i=1}^N \sum_{j=-M}^M a_i b_j \varphi_i(r) \chi_j(\phi(t)) \right| \\ & \leq \max_{\substack{X \in [a,b]^2 \\ t \in (0,T]}} \left| V(X) G(t) - V(X) \sum_{j=-M}^M b_j \chi_j(\phi(t)) \right| + \max_{\substack{X \in [a,b]^2 \\ t \in (0,T]}} \left| V(X) \sum_{j=-M}^M b_j \chi_j(\phi(t)) - \sum_{i=1}^N \sum_{j=-M}^M a_i b_j \varphi_i(r) \chi_j(\phi(t)) \right| \\ & \leq \max_{\substack{X \in [a,b]^2 \\ t \in (0,T]}} |V(X)| \left| G(t) - \sum_{j=-M}^M b_j \chi_j(\phi(t)) \right| + \max_{\substack{X \in [a,b]^2 \\ t \in (0,T]}} \left| \sum_{j=-M}^M b_j \chi_j(\phi(t)) \right| \left| V(X) - \sum_{i=1}^N a_i \chi_j(\phi(t)) \right|, \end{aligned}$$

by assuming $\max_{X \in [a,b]^2} |V(X)| \leq k'$, now by using (8) in the first term and using Lemma 4.1 and relation (30) in the second term on right-hand side in the above inequality and simplifying, we obtain

$$\begin{aligned} & \max_{\substack{X \in [a,b]^2 \\ t \in (0,T]}} \left| V(X) G(t) - \sum_{i=1}^N \sum_{j=-M}^M a_i b_j \varphi_i(r) \chi_j(\phi(t)) \right| \\ & \leq K_1 k' \sqrt{M} e^{-\sqrt{\pi d \beta M}} + 2K_4 K_6 \left(\frac{2}{\pi} (3 + \log(M)) + 1 \right) e^{-\frac{K_8 \sqrt{N}}{b-a}} \leq K_7 \sqrt{M} e^{-K_8 \sqrt{M^*}} \end{aligned}$$

This complete the proof

Now to prove the next theorem. First we need to define the following Sobolev spaces

$$W^{1,2}(\Omega) = H^1(\Omega) = \left\{ w \in L^2(\Omega) : \frac{dw}{dx} \in L^2(\Omega) \right\}.$$

The inner products and norms in $L^2(\Omega)$ are defined as

$$(w, u) = \int_{\Omega} w u d\Omega, \quad \|w\| = (w, w)^{\frac{1}{2}}, \quad \|w\|_1 = (\nabla w, \nabla w)^{\frac{1}{2}},$$

and $H_0^1(\Omega)$ is the space of functions in $H^1(\Omega)$ that vanishes at the boundaries. The Sobolev weighted norm on the $H_0^1(\Omega)$ space is defined as the following

$$\|w\|_{H^1} = \left(\int_{\Omega} (|w|^2 + \Theta |\nabla w|^2) d\Omega \right)^{\frac{1}{2}} = \left(\|w\|^2 + \Theta \|w\|_1^2 \right)^{\frac{1}{2}},$$

where Θ is positive constant.

Theorem 4.2. The solution of Rayleigh-Stokes problem (1)-(3) has been approximated by $\bar{u}(x, y, t)$, using the Sinc-RBF collocation method (22) -(23). If $u^*(x, y, t)$ be the computed solution of the arising systems, and it is supposed that the assumptions of Theorem 4.1 are fulfilled, then the error bound of the Sinc-RBF collocation method satisfies

$$\|u(x, y, t) - u^*(x, y, t)\|_{\infty} \leq K_{SR1} \sqrt{M} e^{-K_{SR2} \sqrt{M^*}},$$

where $M^* = \min\{M, N\}$ and $h = \sqrt{\frac{\pi d}{\mu M}}$.

Proof: Now we can consider the following relation

$$\|u(x, y, t) - u^*(x, y, t)\|_{\infty} \leq \|u(x, y, t) - \bar{u}(x, y, t)\|_{\infty} + \|\bar{u}(x, y, t) - u^*(x, y, t)\|_{\infty}, \tag{32}$$

from Theorem 4.1 we have

$$\|u(x, y, t) - \bar{u}(x, y, t)\|_{\infty} \leq K_7 \sqrt{M} e^{-K_8 \sqrt{M^*}}. \tag{33}$$

For the second term of (32), by substituting $\bar{u}(x, y, t)$ and $u^*(x, y, t)$ in equation (1) and subtracting we have

$$E_1(X) \frac{\partial E_2(t)}{\partial t} = {}_0D_t^{1-\gamma} (E_2(t)) \Delta E_1(X) + E_2(t) \Delta E_1(X) + F_1(x, y, t), \tag{34}$$

where

$$\|\bar{u}(x, y, t) - u^*(x, y, t)\|_{\infty} = E(x, y, t) = E_1(x, y) E_2(t) = E_1(X) E_2(t),$$

and

$$F_1(x, y, t) = |f(x, y, t) - \bar{f}(x, y, t)|.$$

Multiplying both sides of equation (34) by $E(X, t) = E_1(X) E_2(t)$ and integrating over $\Omega \times (0, T]$, we obtain

$$\begin{aligned} \|E_1\|^2 \int_0^T \frac{\partial E_2(t)}{\partial t} E_2(t) dt &= ({}_0D_t^{1-\gamma} E_2(t), E_2(t)) (\Delta E_1(X), E_1(X)) \\ &+ \|E_2\|^2 (\Delta E_1(X), E_1(X)) + (F, E_1(X) E_2(t)), \end{aligned} \tag{35}$$

Since $E_2(t) \in H_0^1$ then $E_2(0) = E_2(T) = 0$ and the left-hand side of equation (35) vanishes, and also

${}_0D_t^{1-\gamma} E_2(t) = {}^C D_t^{1-\gamma} E_2(t)$ then we have

$$({}_0D_t^{1-\gamma} E_2(t), E_2(t)) = \int_0^T \int_0^t \frac{E_2(t) \left(\frac{\partial E_2(\tau)}{\partial \tau} \right)}{(t - \tau)^{1-\gamma}} d\tau dt \leq \frac{T^{\gamma+1}}{\Gamma(\gamma+2)} \|E_2\|_1^2, \tag{36}$$

by substituting (36) in (35) we obtain

$$\begin{aligned} 0 &\leq \frac{T^{\gamma+1}}{\Gamma(\gamma+2)} \|E_2\|_1^2 (\Delta E_1(X), E_1(X)) + \|E_2\|^2 (\Delta E_1(X), E_1(X)) + (F, E_1(X) E_2(t)) \\ &= -\frac{T^{\gamma+1}}{\Gamma(\gamma+2)} \|E_1\|_1^2 \|E_2\|_1^2 - \|E_1\|_1^2 \|E_2\|_1^2 + (F, E_1(X) E_2(t)) \end{aligned} \tag{37}$$

and using the Poincare inequality

$$\|E\| \leq C \|E\|_1,$$

we obtain

$$\frac{1}{C} \|E_1\|^2 \|E_2\|^2 + \frac{T^{\gamma+1}}{\Gamma(\gamma+2)} \|E_1\|_1^2 \|E_2\|_1^2 \leq (F, E_1(X)E_2(t))$$

then

$$\|E_1\|^2 \|E_2\|^2 + \frac{CT^{\gamma+1}}{\Gamma(\gamma+2)} \|E_1\|_1^2 \|E_2\|_1^2 \leq (CF, E_1(X)E_2(t)) \leq \frac{C^2}{2} \|F\|^2 + \frac{1}{2} \|E_1\|^2 \|E_2\|^2$$

Finally we have

$$\|E\|_{H^1} = \left(\|E_1\|^2 \|E_2\|^2 + \Theta \|E_1\|_1^2 \|E_2\|_1^2 \right)^{\frac{1}{2}} \leq C \|F\| \leq K_9 \sqrt{M} e^{-K_{10} \sqrt{M^*}},$$

where $\Theta = \frac{2CT^{\gamma+1}}{\Gamma(\gamma+2)}$. By setting $K_{SR1} = K_7 + K_9$, $K_{SR2} = \min\{K_8, K_{10}\}$, the proof is completed.

5. Numerical Illustration

In this section, we consider the following initial-boundary value problem [13] to test the validity of presented method and to show the efficiency and accuracy of the purposed method.

$$\frac{\partial u(x, y, t)}{\partial t} = {}_0D_t^{1-\gamma} \left[\frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} \right] + \frac{\partial^2 u(x, y, t)}{\partial x^2} + \frac{\partial^2 u(x, y, t)}{\partial y^2} + e^{x+y} \left[(1+\gamma)t^\gamma - 2 \frac{\Gamma(2+\gamma)}{\Gamma(1+2\gamma)} t^{2\gamma} - 2t^{1+\gamma} \right], \quad 0 < t \leq 1, \quad 0 \leq x, y \leq 1,$$

and

$$u(0, y, t) = e^y t^{1+\gamma}, \quad u(1, y, t) = e^{1+y} t^{1+\gamma},$$

$$u(x, 0, t) = e^x t^{1+\gamma}, \quad u(x, 1, t) = e^{1+x} t^{1+\gamma},$$

$$u(x, y, 0) = 0,$$

with the exact solution

$$u(x, y, t) = e^{x+y} t^{1+\gamma}.$$

Sinc-RBF collocation method, that presented in section 4, is applied on the above example with

$$M = 15, d = \frac{3.14}{2}, \mu = \min\{\gamma, 1\}, h = \sqrt{\frac{\pi d}{\mu M}} \text{ and also by choosing various values of}$$

$$h_x = h_y = \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{8} \text{ and with various values of } \gamma = 0.15, 0.5, 0.7, 0.8, 0.9. \text{ The maximum absolute error}$$

in the solution is tabulated in Tables 1 and 2.

Now we define E_∞ for Sinc-RBF collocation method as

$$E_\infty = \max_{1 \leq p \leq n} \max_{1 \leq q \leq n} \max_{-M \leq z \leq M} \left\{ |u(x, y, t) - u^*(x, y, t)| \right\},$$

where $u^*(x, y, t)$ is the approximate solution of $u(x, y, t)$.

Table-1: Error of SE-Sinc-RBF collocation method with $M = 15$

| $h_x = h_y$ | $\gamma = 0.15$ | | $\gamma = 0.5$ | |
|---------------|-------------------------|-----|-------------------------|-----|
| | E_∞ | c | E_∞ | c |
| $\frac{1}{2}$ | 4.8135×10^{-4} | 2 | 3.6910×10^{-3} | 1 |
| $\frac{1}{3}$ | 7.5857×10^{-4} | 2 | 8.2686×10^{-3} | 2.5 |
| $\frac{1}{4}$ | 2.2357×10^{-4} | 1 | 6.7588×10^{-4} | 2 |
| $\frac{1}{5}$ | 9.4872×10^{-5} | 3 | 5.1557×10^{-4} | 2 |
| $\frac{1}{6}$ | 8.6355×10^{-5} | 2 | 5.7606×10^{-5} | 2 |
| $\frac{1}{7}$ | 8.5424×10^{-5} | 3 | 4.9475×10^{-5} | 1.5 |
| $\frac{1}{8}$ | 7.4272×10^{-5} | 3 | 4.5067×10^{-5} | 1 |

Table-2: Error of SE-Sinc-RBF collocation method with $M = 15$

| $h_x = h_y$ | $\gamma = 0.7$ | | $\gamma = 0.8$ | | $\gamma = 0.9$ | |
|---------------|-------------------------|-----|-------------------------|-----|-------------------------|-----|
| | E_∞ | c | E_∞ | c | E_∞ | c |
| $\frac{1}{2}$ | 5.8727×10^{-4} | 3 | 2.4699×10^{-4} | 1 | 4.5673×10^{-3} | 3 |
| $\frac{1}{3}$ | 5.7877×10^{-4} | 3 | 5.2542×10^{-4} | 2 | 6.9153×10^{-4} | 3 |
| $\frac{1}{4}$ | 4.0026×10^{-4} | 1.5 | 3.3831×10^{-4} | 1.5 | 4.5299×10^{-4} | 2.5 |
| $\frac{1}{5}$ | 6.9382×10^{-5} | 1.5 | 2.9326×10^{-4} | 1 | 4.1748×10^{-4} | 1 |
| $\frac{1}{6}$ | 4.4760×10^{-5} | 2.5 | 4.8483×10^{-5} | 1 | 5.9664×10^{-5} | 1 |
| $\frac{1}{7}$ | 4.2741×10^{-5} | 1.5 | 4.4633×10^{-5} | 2 | 4.8253×10^{-5} | 1.5 |
| $\frac{1}{8}$ | 2.2345×10^{-5} | 2 | 3.1920×10^{-5} | 2 | 4.5731×10^{-5} | 1 |

Tables 1 and 2 show that by using our scheme based on RBF and Sinc method with small numbers of basis functions (small values of N and M), one can obtain good results.

Figure-1: (a) shows the graphs of resulting error in the test problem using Sinc-RBF collocation method at $t = \psi(Mh)$ with $\gamma = 0.5, h_x = h_y = \frac{1}{3}, c = 1.5$ and $M=15$.

Figure-1: (b) shows the graphs of resulting error in the test problem using Sinc-RBF collocation method at $t = \psi(Mh)$ with $\gamma = 0.7, h_x = h_y = \frac{1}{4}, c = 1.5$ and $M=15$.

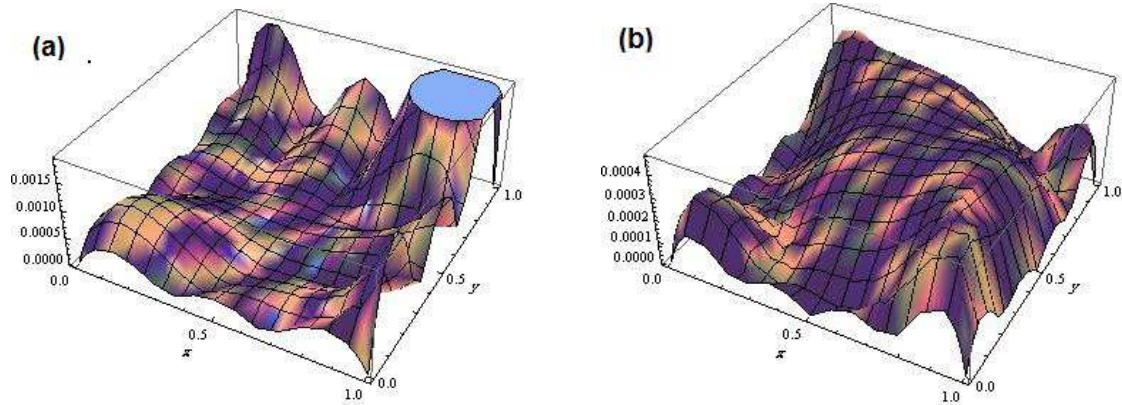


Figure-1: (a) Graphs of resulting error using Sinc-RBF collocation method with $\gamma=0.5, h_x = h_y = \frac{1}{3}, c=1.5$ and $M = 15$. (b) Graphs of resulting error using Sinc-RBF collocation method with $\gamma=0.7, h_x = h_y = \frac{1}{4}, c=1.5$ and $M = 15$.

Conclusion

Our presented method is capable approximating the solution of two-dimensional Rayleigh-Stokes problem with fractional derivative for heated generalized second-grade fluid using a combination of Sinc and RBF methods. This approach is applicable and efficient. Moreover, it can be used with small numbers of basis functions (small values of N and M). Due to the exponentially convergent nature of the method, one can get considerable accurate results. The illustrated example shows the efficiency and accuracy of the work.

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